The Wavelet Method Applied to the Analysis of Evoked Potentials.


The Wavelet

We are using the wavelet transform as a very efficient method for obtaining two-dimensional pictures of a multiscale (an inside) view of sampled signals (EEG potentials).

The general idea of a wavelet used in the wavelet transform of signals (Grossmann and Morlet, 1984; Mallat, 1989; Mallat and Zhong, 1989) comprise all complex square integrable functions \( f(x) \) of finite energy, if their Fourier transform \( \hat{f}(\omega) \) is differentiable and fulfills the admissibility condition, which reads in the normalized form with

\[
c_f = 2\pi \int \left| \frac{\hat{F}(\omega)}{\omega} \right|^2 d\omega < \infty.
\]

This condition essentially means that \( f(x) \) is of zero mean. It implies that \( F(0) = 0, F(\pm \omega_0) = 0 \), and

\[
\int_{-\infty}^{+\infty} f(x) \, dx = 0.
\]

More recently Koendering and van Doorn (1990) recommended using the term »wavelets« for the n-th order derivative of the Gaussian function \( g(x) \) with variance \( \sigma \) and mean value \( \mu \):

\[
g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},
\]

which reads in the normalized form with \( \mu = 0 \):

\[
h(x) = g(x)/g(0) = e^{-x^2/2\sigma^2}.
\]

We selected as the analysing wavelet the negative second-order derivative of the normalized Gaussian function of time \( h(t) \) with respect to \( t \), which is called the »Mexican hat« function:

\[
\psi_{t,\sigma}(t) = -h''(t) = \frac{1}{\sigma^2} \left( 1 - \frac{t^2}{\sigma^2} \right) e^{-\frac{t^2}{\sigma^2}}.
\]

The basic wavelet \( \psi(t) \) in fig. 1, i.e. the unshifted and undilated wavelet function, is given for \( \sigma = 1 \) by

\[
\psi(t) = -h''(t)_{\sigma=1} = \left( 1 - t^2 \right) e^{-t^2/2}.
\]

As for analysing wavelets in general required, eq. (5) and (6) fulfill the admissibility condition (1) and the so-called compatibility condition (2). These two conditions mean that the wavelet should oscillate like a short wave and has no DC-component.

From the chosen analysing wavelet (6) we generated the other members of the wavelet family by translating \( \psi_{t,\sigma}(t) \) by \( \tau \) with \( \tau > 0 \) and dilating \( \psi_{t,\sigma}(t) \) by \( \sigma \), thus obtaining the collection \( \psi_{t,\sigma}(t) \) of the other wavelets

\[
\psi_{t,\sigma}(t) = \psi\left( \frac{t - \tau}{\sigma} \right).
\]

The Transform

The continuous wavelet transform \( W(\tau,\sigma) \) of a real signal \( e(t) \) with respect to the chosen real analysing wavelet \( \psi(t) \) – in general \( \psi(t) \) will be a complex function – is defined by the two-dimensional function

\[
W(\tau,\sigma) = \frac{1}{\sigma^{d/2}} \int_{-\infty}^{+\infty} e(t) \cdot \psi\left( \frac{t - \tau}{\sigma} \right) \, dt
\]

with \( \tau,\sigma \in \mathbb{R} \) and \( \sigma > 0 \), where \( \sigma^{-d/2} \) is the norm in the vector space of measurable, square-integrable d-dimensional functions. In this Hilbert space \( L^2(\mathbb{R}^d) \) eq. (8) is equivalent to the scalar (inner) product of the signal \( e(t) \) with \( \psi_{t,\sigma}(t) \). Each inner product can be interpreted as the convolutions between \( e(t) \) and the set of \( \psi_{t,\sigma}(t) \) in the time domain.

\[
W(\tau,\sigma) = \left\langle e(t) \left| \frac{1}{\sigma^{d/2}} \cdot \psi\left( \frac{t - \tau}{\sigma} \right) \right\rangle = \frac{1}{\sigma^{d/2}} \cdot e(t) * \psi\left( \frac{1}{\sigma} \right)
\]

Hereby is \( t \) the time of the evoked potential \( e(t) \) analysed and \( \tau \) the shift between \( e(t) \) and the wavelet \( \psi_{t,\sigma}(t) \) applied while computing the convolution integral.

The Algorithm

In real (computer) world however, one works with sampled signals obtained from \( e(t) \) by measurements at the instants

\[
t_k = k \Delta t,
\]

where \( 1/\Delta t \) is the sampling frequency (200 Hz). Therefore, eq. (8) must be replaced by its discrete version with \( e[n] \) as the sampled sequence of the EEG potential \( e(t) \).

\[
W(k,\Delta t,\tau,\sigma) = \frac{\Delta t}{\sigma^{d/2}} \cdot \sum_{n=0}^{N-1} e[n] \cdot \psi\left( \frac{(n-k)\Delta t - \tau}{\sigma} \right)
\]

with \( \sigma \in \mathbb{R}^d \) and \( n, k \in \mathbb{Z} \). \( N \) is the number of sampled points in \( e[n] \), which might be upsampled with interpolation by a factor \( \sigma \) (we use \( \sigma = 5 \)). The exponent \( d \) was set at 3. The generation process of the wavelet family was done by translating \( \psi[n] \) by \( \tau = 50–650 \) ms in 1 ms increments, and dilating \( \psi[n] \) by \( \sigma = 20–300 \) ms in 10 ms increments, thus creating 29 wavelet transforms of the sampled evoked potential \( e[n] \).
In our package WaveEEG the discrete wavelet transform is implemented as a fast algorithm of the discrete convolution [5] using the sampled wavelet $\psi[n]$ of half-width $M$ as a convolution filter (kernel), i.e. $\psi[n]$ has $(2M + 1)$ elements with $M < \alpha N/2$. For a symmetric kernel the computer algorithm of the discrete convolution using an upsampled signal $e[n,\alpha]$ with $n = 1,...,\alpha N$ is given by

$$W(k,\Delta\tau,\sigma) = \chi \cdot \sum_{i=0}^{2M} e_{n+i+1-M} \cdot \psi_i$$

(11)

for all $(M + 1 \leq n < \alpha N - M)$, otherwise $W = 0$, by which $\chi$ is a convenient scaling factor.

Contour Plots

In evaluating the W-diagrams we developed a reliable method of computing the statistical significance of the peaks and troughs appearing in the W-diagrams. Using the wavelet transform of 30 different sets of randomly shuffled averaged evoked potentials as a »reference model«, a statistical comparison at the $\pm 3$ s.d. limit was computed. The results were displayed in two-dimensional contour plots (C-plots). A sample C-plot is shown in fig. 4. More informations on this topic and further illustrations are given in [9].

The Equipment

For performing the wavelet analysis, a well-tuned computer equipment with a perfect graphic display and printing capabilities is needed. The computer must be able of processing large-sized arrays of data in a reasonable time. Approximately 48 MByte of (virtual) memory was consumed by this wavelet application.

We used a small-sized VAX cluster (running the virtual multitasking system VMS) configured with a server (VAX-3500), a microVAX-II with two 1 GByte disks, a colour workstation (VAX-3200, running X-Windows), a Macintosh computer (Mac-II), and a PostScript laser printer. All systems were netted together by Ethernet.

The prechecking of optimal algorithms was done on the Macintosh Computer under Mathematica [1] using the excellent GeorgiaTech package SignalProcessing [2]. The final programming for production on the VAX computers was implemented under IDL, the Interactive Data Language [3], a 4-th generation language designed especially for signal processing purposes. This language supports directly the discrete convolution of signals with a kernel, and the Fourier transform. In the last years IDL becomes a standard for high-quality scientific visualization on different computer platforms.

The graphical outputs (plots) were all written in the PostScript language [4]. For binding and resizing the plots into documents we used the portable representation as Encapsulated PostScript (EPS files). This was done on the Macintosh Computer using the Aldus PageMaker, a layout program, and Aldus FreeHand, a graphic tool respectively.

References


